Quantum-Limited Mirror-Motion Estimation

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We experimentally demonstrate optomechanical motion and force measurements near the quantum precision limits set by the quantum Cramér-Rao bounds. Optical beams in coherent and phase-squeezed states are used to measure the motion of a mirror under an external stochastic force. Utilizing optical phase tracking and quantum smoothing techniques, we achieve position, momentum, and force estimation accuracies close to the quantum Cramér-Rao bounds with the coherent state, while the estimation using squeezed states shows clear quantum enhancements beyond the coherent-state bounds.

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The advance of science and technology demands increasingly precise measurements of physical quantities. The probabilistic nature of quantum mechanics represents a fundamental roadblock. Over the last few decades, the issue of quantum limits to precision measurements has been a key driver in the development of quantum measurement theory [1,2]. With the recent technological advances in quantum optical, electrical, atomic, and mechanical systems, quantum limits are now becoming relevant to many metrological applications, such as gravitational-wave detection [3], force sensing [4], magnetometry [5], clocks [6], and biological measurements [7].

It is now recognized that quantum detection and estimation theory [8] provide the appropriate framework for the definition and proof of quantum measurement limits. For parameter estimation and the mean-square error (MSE) criterion, a widely studied quantum limit is the quantum Cramér-Rao bound (QCRB) [8,9]. For gravitational-wave astronomy and many other sensing applications, the estimation of time-varying parameters, commonly called waveforms in the engineering literature, is more relevant. Given the quantum state and dynamics of the sensor and any prior information about the waveform, the QCRBs impose fundamental limits to the waveform estimation accuracy that cannot be violated by any measurement of the sensor [10]. The relevance of the waveform QCRBs to current technology remains an open question, however, as the mathematical formalism provides few clues about when the bounds are attainable, let alone what measurements can approach them in practice.

Experimentally, quantum estimation of an optical phase waveform was recently demonstrated [11,12] using an optical phase tracking method that measures the phase via homodyne detection with feedback control [13], followed by the smoothing of the data [14]. These experiments demonstrate improvements over heterodyne measurements, causal filtering [11], and coherent-state optical beams when squeezed light is used [12], but no comparison with the QCRBs was made, leaving open the question of whether more sophisticated measurement techniques can further improve the estimation accuracy.

In this Letter, we report an experiment that applies the tracking and smoothing techniques to optomechanical motion sensing. We use optical probe beams in coherent and phase-squeezed states to measure the motion of a mirror under an external stochastic force and then compare the smoothing errors with the waveform QCRBs. This is the first time to our knowledge that experimental results have been compared with the waveform QCRBs. Through the comparison, we are able to demonstrate that, remarkably, our measurement method is near optimal in the case of coherent states, and any significant further improvement is ruled out by the QCRBs, despite the large number of optical modes and the endless possibilities of combining and measuring them. The squeezed-state results are further away from the QCRBs but still show clear enhancements over the coherent-state bounds. Despite our focus here on a classical mechanical system, our estimation and optical phase tracking techniques can also be applied to purely quantum systems, in which case measurement backaction is no longer negligible but can be evaded by coherent noise cancellation techniques [15]. Beyond optomechanics, our methods are potentially useful for a wide range of quantum sensing applications [2–7].

Figure 1(a) shows a schematic of our experiment, where the mirror motion is approximated as a mass-spring-damper

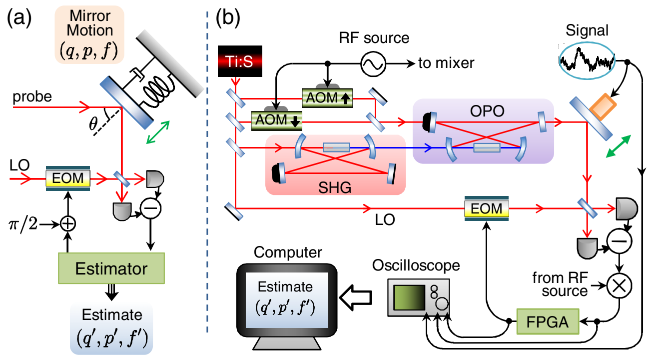

FIG. 1 (color online). (a) Schematic of the mirror-motion estimation. (b) Experimental setup. Local oscillator (LO), radio frequency (RF), titanium sapphire laser (Ti:S), acousto-optic modulator (AOM), electro-optic modulator (EOM), second harmonic generator (SHG), optical parametric oscillator (OPO), field-programmable gate array (FPGA).

system. The mirror, driven by a stochastic force, is illuminated by a probe beam in a coherent state or a phase-squeezed state. The motion of the mirror shifts the phase of the probe beam. We measure this phase shift adaptively by homodyne detection (optical phase tracking) [11–13], and estimate the mirror motion from the optical phase measurements [14].

Optical phase tracking allows us to linearize the measurement results $y(t)$ as

$$y(t) = \varphi(t) + z(t), \tag{1}$$

where $\varphi(t)$ is the optical phase shift and $z(t)$ is a noise term depending on the optical beam statistics [11,12,16]. The phase shift $\varphi(t)$ of the probe beam is caused by the mirror position shift $q(t)$ as

$$\varphi(t) = (2k_0 \cos\theta) q(t), \tag{2}$$

where $k_0 \cos\theta$ is the wave-vector component parallel to the mirror motion and θ is the reflecting angle as shown in Fig. 1(a), fixed at $\pi/4$. We estimate the mirror position $q(t)$, momentum $p(t)$, and external force $f(t)$ from the measurement results $y(t)$. $q(t)$, $p(t)$, $f(t)$, and $z(t)$ are assumed to be zero-mean stationary processes.

Under the linear approximation, the optimal estimate of the mirror position is a weighted sum of the measurement results given by $q'(t) = \int_{-\infty}^{+\infty} d\tau J_q(t-\tau) y(\tau)$, where $J_q(t)$ is a linear filter and prime indicates an estimate. Estimates of momentum $p'(t)$ and external force $f'(t)$ are similarly defined. The integration limits are approximated as $\pm\infty$ because we use data long before and after t to obtain the estimates at the intermediate time t via smoothing [14]. The optimal position filter $J_q(t)$ is obtained by minimizing the MSE $\Pi_q = \langle [q'(t) - q(t)]^2 \rangle$, which is averaged over the probability measures for $z(t)$ and $q(t)$ (Π_p and Π_f are similarly defined). The optimal filters and the minimum MSEs are calculated by moving to the frequency domain [16]. The minimum MSEs $\Pi_x^{\min}$ $(x = q, p, f)$ are given by [14,16,17]

$$\Pi_x^{\min} = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \left(\frac{1}{S_x(\omega)} + \frac{|g_{\varphi x}(\omega)|^2}{S_z(\omega)} \right)^{-1}, \tag{3}$$

where $S_x(\omega)$ $(x = q, p, f, z)$ is a spectral density defined as $S_x(\omega) := \int_{-\infty}^{+\infty} d\tau \langle x(t) x(t+\tau) \rangle e^{i\omega\tau}$, and $g_{\varphi x}(\omega)$ is a transfer function that relates the optical phase shift φ to the target variables $(x = q, p, f)$ by $\tilde{\varphi}(\omega) = g_{\varphi x}(\omega) \tilde{x}(\omega)$, with the tilde indicating a Fourier transform.

We now consider the QCRBs on the MSEs. The waveform QCRBs are derived from the quantum properties of the probe beams and prior statistics of the target system (mirror motion) and do not depend on the measurement and postprocessing method. The QCRBs for our situation are [10]

$$\Pi_x \geq \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \left(\frac{1}{S_x(\omega)} + |g_{\varphi x}(\omega)|^2 4 S_{\Delta I}(\omega) \right)^{-1}, \tag{4}$$

where $S_{\Delta I}(\omega)$ is the spectral density of the probe-beam photon flux. Comparing Eq. (3) with Eq. (4), we find that our method is not only optimal for the given measurement in the context of classical statistics, but can also reach the QCRBs valid for any measurement if $4S_{\Delta I}(\omega) = 1/S_z(\omega)$. This means that, to attain the QCRBs, (i) the probe beam should be in a minimum-uncertainty state with respect to the phase and the photon flux, and (ii) the measurement noise $z(t)$ should consist of intrinsic phase noise only.

Our experiment uses broadband phase-squeezed states, including coherent states as the small-squeezing limit. The noise term $z(t)$ in the normalized homodyne outputs can be written in a quadratic approximation [12,16] as

$$\langle z(t) z(\tau) \rangle = \frac{\bar{R}_{\mathrm{sq}}}{4|\alpha|^2} \delta(t-\tau), \tag{5}$$

$$\bar{R}_{\mathrm{sq}} = \sigma_\varphi^2 e^{2r_p} + (1 - \sigma_\varphi^2) e^{-2r_m}, \tag{6}$$

where $r_m (r_p)$ is the squeezing (antisqueezing) parameter $(r_p \geq r_m \geq 0)$, α is the coherent amplitude of the probe beam, and σ_φ^2 is the steady-state MSE of the optical phase estimate in the real-time feedback loop $(\sigma_\varphi^2 \ll 1)$. $\bar{R}_{\mathrm{sq}}$ is called the *effective squeezing factor* [12], which takes into account the antisqueezed amplitude quadrature as well as the squeezed phase quadrature. The noise spectral density $S_z(\omega)$ and the photon-flux spectral density $S_{\Delta I}(\omega)$ are [16]

$$S_z(\omega) = \frac{\bar{R}_{\mathrm{sq}}}{4|\alpha|^2}, \qquad S_{\Delta I}(\omega) \approx |\alpha|^2 e^{2r_p}. \tag{7}$$

Here we assume that the bandwidth of squeezing is broad compared to the bandwidth of system parameters, but not too large so that the photon-flux fluctuations do not diverge (see the Supplemental Material [16]).

The necessary condition to reach the QCRBs is now given by $e^{2r_p} = 1/\bar{R}_{\mathrm{sq}}$. For coherent states $(r_m = r_p = 0$

and $\bar{R}_{sq} = 1$), this condition is always satisfied, so QCRB-limited estimation is possible within the quadratic approximation. On the other hand, the squeezed-state QCRB is attainable only if (i) the squeezed state is pure ($e^{2r_p} = e^{2r_m}$) and (ii) the optical phase tracking works well enough such that $\sigma_\varphi^2 \simeq 0$. Thus, in a real experimental situation, the squeezed-state QCRB is more difficult to reach than the coherent-state QCRB. We emphasize, however, that our estimation results are still comparable to the squeezed-state QCRBs and better than the coherent-state bounds.

Figure 1(b) shows our experimental setup. A continuous-wave titanium sapphire laser is used as a light source at 860 nm. Phase-squeezed states are generated by an OPO [12,18]. The OPO is driven below threshold by a 430 nm pump beam. Optical sidebands at ± 5 MHz are used as a carrier beam generated by acousto-optic modulators [11,12]. To avoid experimental complexities, the pump power is fixed at 80 mW, producing squeezing and antisqueezing levels of -3.62 ± 0.26 and 6.00 ± 0.15 dB. The effective squeezing factor $\bar{R}_{sq}$ varies from -3.28 to -3.48 dB depending on the probe amplitude. To make a coherent state, we simply block the pump beam.

A mirror (12.7 mm in diameter, 1.5 mm in thickness, 0.444 g in weight) is attached to a piezoelectric transducer (PZT) weighing 0.432 g. We assume the mass of this PZT-mounted mirror to be $m = (0.444 + 0.432/3)$ g $= 5.88 \times 10^{-4}$ kg from the uniformity of the PZT [16]. The transfer function of the PZT-mounted mirror (the relation of applied voltage to actual position shift) is measured before the estimation experiments. We use this transfer function to construct optimal filters and calculate the QCRBs [16].

In the estimation experiments, the PZT-mounted mirror is driven by an Ornstein-Uhlenbeck process. This signal is generated by a random signal generator followed by a low-pass filter with a cutoff frequency of $\lambda = 5.84 \times 10^4$ rad/s. We drive the PZT within the linear response range so that the external force $f(t)$ is proportional to the signal. Thus the external force $f(t)$ is also an Ornstein-Uhlenbeck process given by

$$\frac{df(t)}{dt} = -\lambda f(t) + w(t), \tag{8}$$

where $w(t)$ is a zero-mean white Gaussian noise satisfying $\langle w(t)w(\tau)\rangle = \kappa\delta(t-\tau)$. In the experiment, we set $\kappa = 1.67 \times 10^3$ N^2 s^{-1}.

A fraction of the laser beam is used as a local oscillator beam, which is optically mixed with the probe beam at a 1:1 beam splitter for homodyne detection. The overall efficiency of the detection is 87% [16]. The homodyne output is demodulated and recorded with an oscilloscope. The measured data are postprocessed using a computer to produce the estimates. The demodulated homodyne output is also processed by a FPGA for the real-time feedback based on Kalman filtering, which approximates the mirror motion as a mass-spring-damper system [14]. Note that we use this approximate model only for the real-time feedback, not for the estimation. In the experiment, we have another low-gain, low-frequency feedback loop to prevent environmental phase drift.

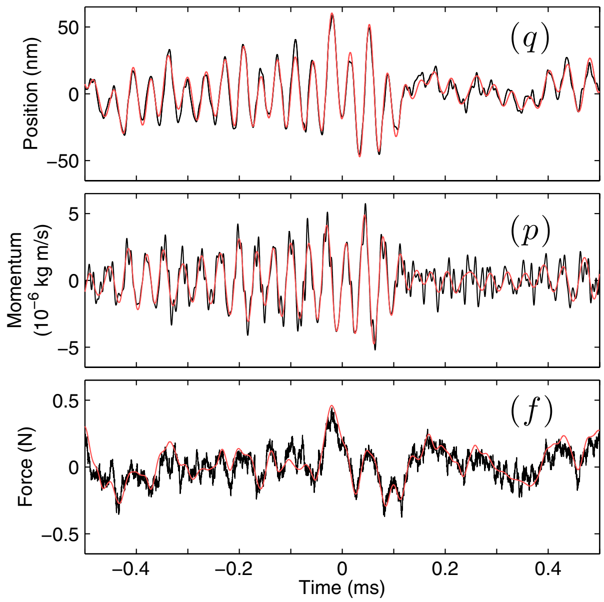

FIG. 2 (color online). Time-domain results for (q) position, (p) momentum, and (f) external force, respectively, with $|\alpha|^2 = 6.24 \times 10^6$ s^{-1} and the probe beam in a phase-squeezed state. The black lines are the signals to be estimated. The red lines (gray lines in print) are the estimates.

Figure 2 shows one of the time-domain results for the mirror-motion estimation with phase-squeezed states. The black lines are the signals to be estimated (for the evaluation, see the Supplemental Material [16]). The external force f is an Ornstein-Uhlenbeck process given by Eq. (8). The periodic oscillations of q and p arise from the mechanical resonance of the PZT-mounted mirror, the frequency of which is 1.76×10^5 rad/s [16]. The red lines are the estimates, which agree well with the signals. This 1 ms long data are obtained with a sampling frequency of 10 MHz, and are repeated 300 times to evaluate the MSEs.

We perform mirror-motion estimation with probe beams in the coherent state and the phase-squeezed state, each with four different amplitudes. Figure 3 shows the $|\alpha|^2$ dependence of the MSEs of the position, momentum, and external force estimation. Figure 3 shows three key results. The first key result: Experimental results agree well with the theoretical predictions [traces (i) and (iii)]. The small discrepancies may be attributed to the low-frequency noise due to environmental phase drift and slight changes of the mirror properties (e.g., the resonant frequency) during the experiment. The second key result: The experimental results are close to the waveform QCRBs. In particular, the experimental results for coherent states (green circles) are very close to the coherent-state QCRBs [traces (ii)]. The closeness (i.e., relative differences between the

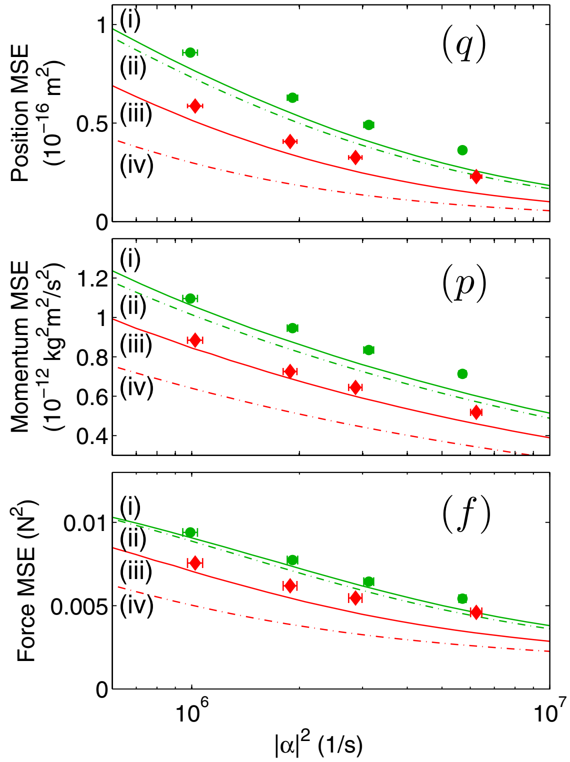

FIG. 3 (color online). Experimental and theoretical MSEs of the (q) position, (p) momentum, and (f) external force, plotted versus the probe amplitude squared, $|\alpha|^2$. The green circles are the results for coherent states, and the red diamonds are those for phase-squeezed states. The green solid curves [traces (i)] are simulated prediction curves of the estimates, which were calculated by considering the experimental imperfections. The green dot-dashed curves [traces (ii)] are the coherent-state QCRBs. The red solid lines [traces (iii)] are the simulated prediction curves for a phase-squeezed probe beam, where we use the quadratic approximation as in Ref. [12]. The red dot-dashed curves [traces (iv)] are the squeezed-state QCRBs.

experimental MSEs and the coherent-state QCRBs) is quantified as $28 \pm 12\%$, $15 \pm 6\%$, and $11 \pm 6\%$ on average for the position, momentum, and force estimates, respectively. The small differences between the prediction curves [traces (i)] and the coherent-state QCRBs [traces (ii)] are attributed to the imperfect detection efficiency. The experimental results of squeezed states (red diamonds) are also comparable to the squeezed-state QCRBs [traces (iv)], although the gaps are larger due to the impurity of the squeezed states. The third key result: The experimental results for squeezed states show clear quantum enhancement, mostly overcoming the coherent-state QCRBs. The quantum enhancements (i.e., relative reduction of MSEs compared to the coherent-state QCRBs) are quantified as $15 \pm 8\%$ and $12 \pm 2\%$ on average for the position and momentum estimates, respectively. The force estimate at the highest probe amplitude is slightly worse than the coherent-state QCRB, which should be due to the low-frequency noise from the environment. Note that we still observe quantum enhancement of the force estimation (except the estimate at the highest probe amplitude), which is quantified as $12 \pm 2\%$ on average.

In conclusion, we have experimentally demonstrated quantum-limited mirror-motion estimation via optical phase tracking. Our experiment reveals that the coherent-state QCRB is almost attainable by our experimental method. Although the squeezed-state QCRB turns out to be more difficult to reach because of the impurity of the squeezed states, quantum enhancement beyond the coherent-state QCRB is clearly observed. These results demonstrate the potential of our theoretical and experimental methods for future quantum metrological applications.

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